A Sequent Calculus for Orthologic

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May 21, 2001

1 Introduction

Quantum Logic is motivated semantically by considering non-Boolean lattices which derive non-classical logics. Once this is done it is natural to attempt to find syntactical systems for deriving these non-classical logical validities. Chiara and Giuntini[1] give an example of such a system for orthologic. The details of the proofs of soundness and completeness are not given in their paper.

In this paper a modified system is shown to be sound and complete. Completeness is proven by showing that the new system derives the rules for Chiara and Giuntini's system, and then showing that Chiara and Giuntini's system is complete. The modified system is simpler than Chiara and Giuntini's, and reduces the need for cuts in derivations.

2 The Original Deduction System

(OL1) $T \cup \{\alpha\} \vdash \alpha$

(OL2) $$\frac{T \vdash \alpha \qquad T' \cup \{\alpha\} \vdash \beta}{T \cup T' \vdash \beta}$$

(OL3) $T \cup \{\alpha \wedge \beta\} \vdash \alpha$

(OL4) $T \cup \{\alpha \wedge \beta\} \vdash \beta$

(OL5) $$\frac{T \vdash \alpha \qquad T \vdash \beta}{T \vdash \alpha \wedge \beta}$$

(OL6) $$\frac{T \cup \{\alpha, \beta\} \vdash \gamma}{T \cup \{\alpha \wedge \beta\} \vdash \gamma}$$

(OL7) $$\frac{\{\alpha\} \vdash \beta \qquad \{\alpha\} \vdash \neg\beta}{\vdash \neg\alpha}$$

(OL8) $T \cup \{\alpha\} \vdash \neg\neg\alpha$

(OL9) $T \cup \{\neg\neg\alpha\} \vdash \alpha$

(OL10) $T \cup \{\alpha \wedge \neg\neg\alpha\} \vdash \beta$

(OL11) $$\frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$$

Figure 1: Chiara and Giuntini's deduction system for orthologic

The deductive system OL (Figure 1) given in chapter 6 of Chiara and Giuntini is a of combination of a sequent calculus and a natural deduction system. The system is like a natural deduction system because it has both introduction and elimination rules. It is also like a sequent calculus system because it has both left introduction and right introduction rules.

In a sequent calculus system the left introduction rules take the place of elimination rules so one of these sets of rules is redundant. Because the system more resembles a sequent calculus, the new system given in this paper modifies Chiara and Giuntini's making it more like a sequent calculus.

$$\text{OL6}\,\frac{T \cup \{\alpha, \beta\} \vdash \alpha}{T \cup \{\alpha \wedge \beta\} \vdash \alpha} \qquad \text{OL6}\,\frac{T \cup \{\alpha, \beta\} \vdash \beta}{T \cup \{\alpha \wedge \beta\} \vdash \beta}$$

Figure 2: Derivation of OL3 and OL4

The elimination rules are removed because they can be derived from OL1 and OL6. This derivation is shown in Figure 2.

Removing OL7 is another simplification to the deduction system. This rule is never used in the proof of completeness, and removing it maintains soundness.

A property that sequent calculi have is that formulas become more complicated as the derivation gets deeper. To capture this, and to reduce the number of rules needed, the rules governing ¬ have been changed.

3 A New Deduction System

(OL′1) $T \cup \{\alpha\} \vdash \alpha$

(OL′2) $\dfrac{T \vdash \alpha \qquad T \vdash \beta}{T \vdash \alpha \wedge \beta}$

(OL′3) $\dfrac{T \cup \{\alpha, \beta\} \vdash \gamma}{T \cup \{\alpha \wedge \beta\} \vdash \gamma}$

(OL′4) $\dfrac{T \vdash \alpha}{T \cup \{\neg\alpha\} \vdash \beta}$

(OL′5) $\dfrac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$

(OL′6) $\dfrac{T \vdash \alpha}{T \vdash \neg\neg\alpha}$

(OL′7) $\dfrac{T \cup \{\alpha\} \vdash \beta}{T \cup \{\neg\neg\alpha\} \vdash \beta}$

(OL′Cut) $\dfrac{T \vdash \alpha \qquad T' \cup \{\alpha\} \vdash \beta}{T \cup T' \vdash \beta}$

Figure 3: New deduction system for orthologic OL′

(OL8) $$\text{OL}'6\,\frac{T \cup \{\alpha\} \vdash \alpha}{T \cup \{\alpha\} \vdash \neg\neg\alpha}$$

(OL9) $$\text{OL}'7\,\frac{T \cup \{\alpha\} \vdash \alpha}{T \cup \{\neg\neg\alpha\} \vdash \alpha}$$

(OL10) $$\text{OL}'3\,\frac{\text{OL}'4\,\dfrac{T \cup \{\alpha\} \vdash \alpha}{T \cup \{\alpha, \neg\alpha\} \vdash \beta}}{T \cup \{\alpha \wedge \neg\alpha\} \vdash \beta}$$

Figure 4: Derivation of OL8 - OL10

A modified deduction system is given in Figure 3. Most of the deductions given in OL are either in OL′ or are derivable in this new system. OL1, OL2, OL5, OL6, and OL11 are included in OL′. Since OL1 and OL6 are included, OL3 and OL4 are derivable as shown in the previous section. OL7 is not derivable but was not needed. The derivations of OL8, OL9, and OL10 are given in Figure 4.

4 Soundness of OL′

This section shows that the system OL′ is sound with respect to Kripkean semantics of orthologic. The axioms of OL′ will be shown to be valid, and for each rule of inference the conclusion will be shown to be valid under the assumption that the hypotheses are valid. Then a simple inductive argument shows that the conclusion of any derivation is valid.

4.1 OL′1

Let i be a world of K such that $i \models T \cup \{\alpha\}$. Then $i \models \alpha$. So $T \cup \{\alpha\} \models \alpha$.

4.2 OL′2

Suppose K is such that $T \models \alpha$ and $T \models \beta$. Let i be a world of K such that $i \models T$. Then $i \models \alpha$ and $i \models \beta$. By lemma 2.3.2 from Chiara and Giuntini, $i \models \alpha \wedge \beta$. So $T \models \alpha \wedge \beta$.

4.3 OL′3

Suppose K is such that $T \cup \{\alpha, \beta\} \models \gamma$. Let i be a world of K such that $i \models T \cup \{\alpha \wedge \beta\}$. So $i \models T$. Also $i \models \alpha \wedge \beta$. By lemma 2.3.2 from Chiara and Giuntini, $i \models \alpha$ and $i \models \beta$. Therefore $i \models T \cup \{\alpha, \beta\}$. By assumption $i \models \gamma$. So $T \cup \{\alpha \wedge \beta\} \models \gamma$.

4.4 OL$'$4

Suppose K is such that $T \models \alpha$. Let i be a world of K such that $i \models T \cup \{\neg\alpha\}$. Therefore $i \models T$ and by the assumption $i \models \alpha$. But also $i \models \neg\alpha$. By lemma 2.1.1 from Chiara and Giuntini, $\forall j \not\perp i (j \not\models \alpha)$. In particular since $i \not\perp i$, then $i \not\models \alpha$. This is a contradiction. There are no worlds of K that model $T \cup \{\neg\alpha\}$. Therefore if i were such a world then $i \models \gamma$ vacuously. So $T \cup \{\neg\alpha\} \models \gamma$.

4.5 OL$'$5

Suppose K is such that $\{\alpha\} \models \beta$. Let i be a world of K such that $i \models \neg\beta$. Let j be such that $j \not\perp i$. By lemma 2.3.1 from Chiara and Giuntini, $j \not\models \beta$. Suppose that $j \models \alpha$. Then by assumption $j \models \beta$, which is a contradiction. Therefore $j \not\models \alpha$. So $i \models \neg\alpha$ and $\{\neg\beta\} \models \neg\alpha$.

4.6 OL$'$6

Suppose K is such that $T \models \alpha$. Let i be a world of K such that $i \models T$. Therefore $i \models \alpha$. Let j be such that $j \not\perp i$. Suppose $j \models \neg\alpha$. Then forall $k \not\perp i$, $k \not\models \alpha$. In particular $i \not\models \alpha$. This is a contradiction. Therefore $j \not\models \neg\alpha$ and then $i \models \neg\neg\alpha$. So $T \models \neg\neg\alpha$

4.7 OL$'$7

Suppose K is such that $T \cup \{\alpha\} \models \beta$. Let i be a world of K such that $i \models T \cup \{\neg\neg\alpha\}$. Let j be such that $j \not\perp i$. By lemma 2.3.1 from Chiara and Giuntini, $j \not\models \neg\alpha$. So by lemma 2.3.1 there is some $k \not\perp j$ such that $k \models \alpha$. Therefore by theorem 2.1 from Chiara and Giuntini $i \models \alpha$. So $i \models T \cup \{\alpha\}$, and by assumption $i \models \beta$. Therefore $T \cup \{\neg\neg\alpha\} \models \beta$.

4.8 OL$'$Cut

Suppose K is such that $T \models \alpha$ and $T' \cup \{\alpha\} \models \beta$. Let i be a world of K such that $i \models T \cup T'$. Since $i \models T$, then $i \models \alpha$. So $i \models T' \cup \{\alpha\}$. Therefore $i \models \beta$. So $T \cup T' \models \beta$.

5 Completeness of OL$'$

Since all rules of OL have been shown to be derivable from OL$'$, except for OL7, it is sufficient to show that OL less OL7 is complete. This section proves completeness with respect to Kripkean semantics in exactly the same way Chiara and Giuntini do in chapter 6. The details missing from that proof is given here.

5.1 Definition of the Canonical models

The canonical model for syntactical deduction is $K =< I, R, \Pi, \rho >$ satisfying the following.

1. I is the set of consistent deductively closed sets of formulas.
2. Rij if $\forall\alpha(i \vdash \alpha \Rightarrow j \nvdash \neg\alpha)$.
3. Π is the complete set of propositions from the frame $< I, R >$.
4. $\rho(p) = \{i \in I | p \in i\}$.

To show this is an acceptable model, one needs to show that R is reflexive and symmetric and $\rho(p) \in \Pi$.

Suppose R were not reflexive. Then for some $i \in I$ and some α, $i \vdash \alpha$ and $i \vdash \neg\alpha$. By OL5, $i \vdash \alpha \wedge \neg\alpha$. By OL2 and OL 10, $i \vdash \beta$ for any β. This means i is inconsistent, contradicting the fact that $i \in I$. So R must be reflexive.

Suppose Rij. Then $\forall i \in I\ i \vdash \alpha \Rightarrow j \nvdash \neg\alpha$. Suppose $j \vdash \alpha$. By OL8 and OL2, $j \vdash \neg\neg\alpha$. Therefore $j \nvdash \neg\alpha$. Thus Rji and R is symmetric.

Let $i \notin \rho(p)$. By lemma 2.1.1 from Chiara and Giuntini, if one shows that there is some $j \not\perp i$ and $j \perp \rho(p)$, then $\rho(p)$ is a proposition in $< I, R >$. Since $i \notin \rho(p)$, then $p \notin i$. So $i \nvdash p$. $i \nvdash \neg\neg p$ because otherwise by OL9 and OL2 one would get $i \vdash p$. By the weak Lindenbaum theorem, there is a T such that $i \not\perp T$ and $T \vdash \neg p$. Let $j = \overline{T}$. Then $\neg p \in j$ and $i \not\perp j$. Let $k \in \rho(p)$. Then $p \in k$ and $k \vdash p$. Since $j \vdash \neg p$, then $j \perp k$. So $j \perp \rho(p)$ as needed.

5.2 Lemma 6.4 $i \models \alpha \Leftrightarrow \alpha \in i$

Proof by induction on the structure of α.

If $\alpha = p$ then $i \models p$ iff $i \in \rho(p)$ iff $p \in i$.

Suppose $\alpha = \neg\beta$ and $\neg\beta \in i$. So $i \vdash \neg\beta$. Suppose $j \in \rho(\beta)$. Then by induction $\beta \in j$ and $j \vdash \beta$. Then $i \perp j$ because they are syntactically incompatible. Therefore $i \in \rho(\beta)'$. so $i \models \neg\beta$.

Now suppose $\neg\beta \notin i$. Then $i \nvdash \neg\beta$. By the Weak Lindenbaum theorem, there is a T such that $i \not\perp T$, and $T \vdash \beta$. Let $j = \overline{T}$. Therefore $i \not\perp j$. Since $T \vdash \beta$, then $j \vdash \beta$. By lemma 2.3.1 from Chiara and Giuntini, $i \nvDash \neg\beta$.

Suppose $\alpha = \beta \wedge \gamma$ and $\beta \wedge \gamma \in i$. So $i \vdash \beta \wedge \gamma$. By OL3 and OL2, $i \vdash \beta$. By OL4 and OL2, $i \vdash \gamma$. So $\{\beta, \gamma\} \subseteq i$. By inductive hypothesis, $i \models \{\beta, \gamma\}$. By lemma 2.3.2 from Chiara and Giuntini, $i \models \beta \wedge \gamma$.

Now suppose $i \models \beta \wedge \gamma$. By lemma 2.3.2 from Chiara and Giuntini, $i \models \{\beta, \gamma\}$. By inductive hypothesis, $\{\beta, \gamma\} \subseteq i$. So $i \vdash \beta$ and $i \vdash \gamma$. By OL5, $i \vdash \beta \wedge \gamma$; so $\beta \wedge \gamma \in i$. This completes the lemma.

5.3 Completeness: If $T \nvdash \alpha$ then $T \nvDash \alpha$

Suppose $T \nvdash \alpha$. T must be consistent because if it were the case that $T \vdash \beta$ and $T \vdash \neg\beta$ then one could conclude that $T \vdash \beta \wedge \neg\beta$. However $\beta \wedge \neg\beta \vdash \alpha$

by OL10. By OL2 $T \vdash \alpha$, which is a contradiction. Let $i = \overline{T}$. $i \in I$ since $\overline{T}$ is consistent and deductively closed. By lemma 6.4 from Chiara and Giuntini, $i \models T$ since $T \subseteq i$. Suppose that $i \models \alpha$. Then by lemma 6.4 from Chiara and Giuntini, $\alpha \in i$. Therefore $T \vdash \alpha$ by definition of i. This is a contradiction; so $i \not\models \alpha$. So $T \not\models \alpha$.

6 Conclusion

System OL given by Chiara and Giuntini can be simplified and made more like a sequent calculus system. OL$'$ is an example of such a system. OL was shown to be sound and complete with respect to Kripkean semantics of orthologic. The strong and weak double negations rules were changed so that cuts would not been needed to use them. An attempt was made to create a system without a need for cut. Such a system would lead directly to an algorithm from deciding if a sequent was valid or not[2]. Unfortunately there are still some valid sequents that require cut in their derivation. More work needs to be done to find a cut free sequent calculus for orthologic.

References

[1] Maria Lusia, Dalla Chiara, and Roberto Giuntini. Quantum logics, March 2000.

[2] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 1996.