Math is Functional Programming

Implementing Real Numbers

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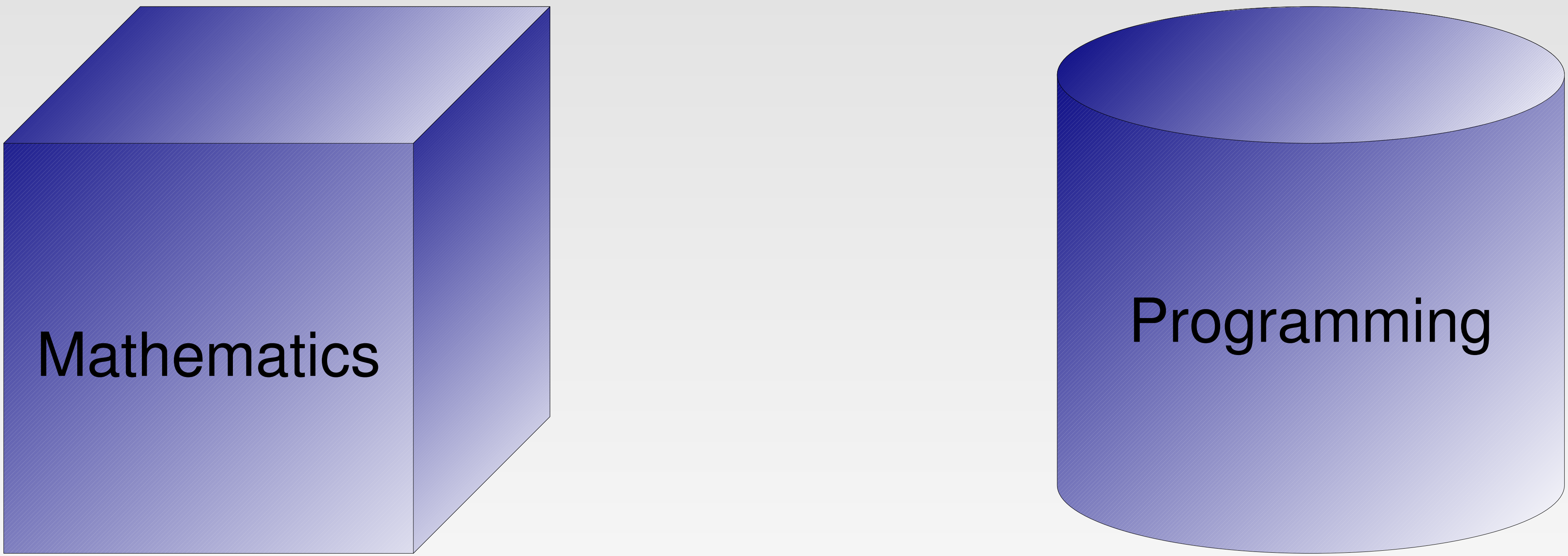
Functioneel Programmeren dag 2008

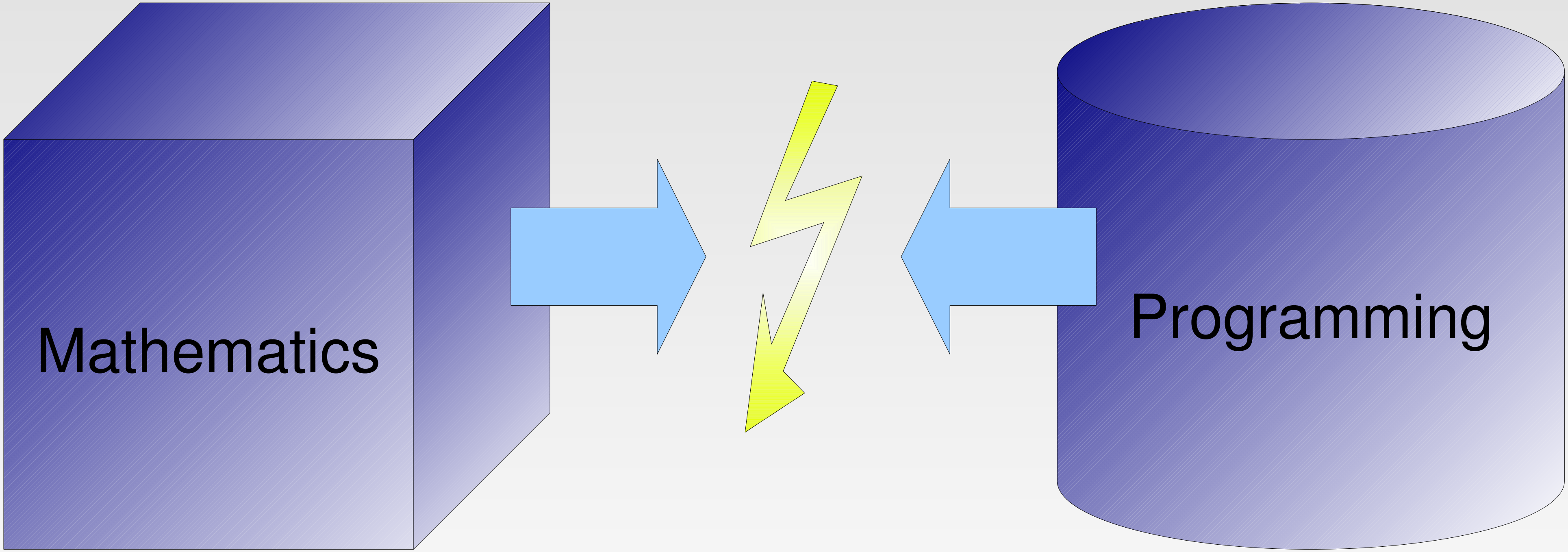
2008-01-11

Takahashi method

A Tale of Two Cities



A Tale of Two Cities



Types

Types

$$A \Rightarrow B$$

```
a  ->  b
```

Types

$$A \times B$$

`(a, b)`

Types

$$A + B$$

`Either a b`

Types

$$1$$

$$()$$

Types

$$\mathbb{N}$$

```
data Nat = O | S Nat
```

Dependent Types

Dependent Types

```
Vec :: Nat -> * -> *
Vec O _ = ()
Vec (S n) a = (a, Vec n a)
```

Dependent Types

$$B : A \Rightarrow \star$$

$$\Pi a{:}A.\ (B\ a)$$

Dependent Types

```
vZero : 
```
$$\mathrm{vZero} : \prod n{:}\mathbb{N}.\ \mathrm{Vec}\ n\ \mathbb{N}$$

```
vZero O = ()
vZero (S n) = (O, vZero n)
```

Dependent Types

`vAppend :`

$$\prod n{:}\mathbb{N}.\ \mathrm{Vec}\ n\ \mathbb{Q} \Rightarrow \prod m{:}\mathbb{N}.\ \mathrm{Vec}\ m\ \mathbb{Q} \Rightarrow \mathrm{Vec}\ (n+m)\ \mathbb{Q}$$

Dependent Types

$$B : A \Rightarrow \star$$

$$\Sigma a{:}A.\ (B\ a)$$

Dependent Types

```
data Country = CA | US
newType ZipCode = ...
newType PostalCode = ...
```

Dependent Types

```
PostalType :: Country -> *
PostalType CA = PostalCode
PostalType US = ZipCode
```

Dependent Types

$$\sum c{:}\mathrm{Country}.\ \mathrm{PostalType}\ c$$

`(CA, read “K1A 0B1”)`

Logic

Logic

$$A \wedge B$$

$$A \times B$$

Logic

$$A \Rightarrow B$$

$$A \Rightarrow B$$

Logic

$$A \vee B$$

$$A + B$$

Logic

$$\top$$

$$1$$

Logic

$$\perp$$

$$0$$

```
data Void
```

Logic

$$\neg A := A \Rightarrow \bot$$

Logic

$$((),()) : \top \wedge \top$$

$$((),()) : 1 \times 1$$

$$((),()) : ((),())$$

Logic

$$\texttt{left ()} : \top \vee \bot$$

$$\texttt{left ()} : 1 + 0$$

```
left () : Either () Void
```

Logic

$$\mathtt{id} : A \Rightarrow A$$

```
id : a -> a
```

Logic

$$\texttt{abort}: \bot \Rightarrow A$$

```
abort H = case H of {}
```

Logic

$$((),??) : \top \wedge \bot$$

Logic

`?? : ⊥`

Inconsistency

```
fix id: ⊥
```

Consistency

~~General Recursion~~

Consistency

Structural Recursion

Structural Recursion

```
Vec :: Nat -> * -> *
Vec O _ = ()
Vec (S n) a = (a, Vec n a)
```

Structural Recursion

All Functions Terminate

Decidable Typechecking

Mathematics

Mathematics

$$\mathbb{N}$$

Mathematics

$$(+) : \mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \mathbb{N}$$

$$\begin{aligned} \mathrm{O} \quad + m &:= m \\ \mathrm{S}\, n + m &:= \mathrm{S}\,(n + m) \end{aligned}$$

Mathematics

$$(==) : \mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \mathrm{Bool}$$

$$\begin{aligned}
\mathrm{O} &== \mathrm{O} &&:= \mathrm{True} \\
\mathrm{S}\ n &== \mathrm{S}\ m &&:= n == m \\
_ &== _ &&:= \mathrm{False}
\end{aligned}$$

Mathematics

$$\langle \cdot \rangle : \mathrm{Bool} \Rightarrow \bigstar$$
$$\langle \mathrm{True} \rangle := \top$$
$$\langle \mathrm{False} \rangle := \bot$$

Mathematics

$$() : \langle 2+2 == 4 \rangle$$

$$() : \top$$

Mathematics

$$?? : \langle 0 == 1 \rangle$$

$$?? : \bot$$

Quantifiers

$$\forall a{:}A.\ (B\ a)$$

$$\Pi a{:}A.\ (B\ a)$$

Quantifiers

$$\exists a{:}A.\ (B\ a)$$

$$\sum a{:}A.\ (B\ a)$$

Lemma 1

$$\begin{array}{lll} \text{lemma1} & : & \forall n{:}\mathbb{N}.\ \langle n == n + \mathrm{O} \rangle \\ \text{lemma1 O} & := & () \\ \text{lemma1 (S } n) & := & \text{lemma1 } n \end{array}$$

Lemma 1

$$\mathrm{lemma1}\ \mathrm{O} : \langle \mathrm{O} == \mathrm{O} + \mathrm{O} \rangle$$

$$\mathrm{lemma1}\ \mathrm{O} : \top$$

$$() : \top$$

Lemma 1

$$\mathrm{lemma1}\ \mathrm{O} : \langle \mathrm{O} == \mathrm{O} + \mathrm{O} \rangle$$

$$\mathrm{lemma1}\ \mathrm{O} : \top$$

$$() : \top$$

Lemma 1

$$
\begin{aligned}
\mathrm{lemma1}\ \mathrm{O} &: \langle \mathrm{O} == \mathrm{O} + \mathrm{O} \rangle \\
\mathrm{lemma1}\ \mathrm{O} &: \top \\
() &: \top
\end{aligned}
$$

Lemma 1

$$\text{lemma1}\ (\mathrm{S}\ n) : \langle \mathrm{S}\ n == \mathrm{S}\ n + \mathrm{O} \rangle$$

$$\text{lemma1}\ (\mathrm{S}\ n) : \langle \mathrm{S}\ n == \mathrm{S}\ (n + \mathrm{O}) \rangle$$

$$\text{lemma1}\ (\mathrm{S}\ n) : \langle n == n + \mathrm{O} \rangle$$

$$\text{lemma1}\ n : \langle n == n + \mathrm{O} \rangle$$

Lemma 1

$$\mathrm{lemma1}\ (\mathrm{S}\ n) : \langle \mathrm{S}\ n == \mathrm{S}\ n + \mathrm{O} \rangle$$
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Lemma 1

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Lemma 1

$$\mathrm{lemma1}\ (\mathrm{S}\ n) : \langle \mathrm{S}\ n == \mathrm{S}\ n + \mathrm{O} \rangle$$
$$\mathrm{lemma1}\ (\mathrm{S}\ n) : \langle \mathrm{S}\ n == \mathrm{S}\ (n + \mathrm{O}) \rangle$$
$$\mathrm{lemma1}\ (\mathrm{S}\ n) : \langle n == n + \mathrm{O} \rangle$$
$$\mathrm{lemma1}\ n : \langle n == n + \mathrm{O} \rangle$$

Lemma 2

$\text{lemma2} : \forall n, m{:}\mathbb{N}.\ \langle n == m \rangle \Rightarrow \langle m == n \rangle$

$$
\begin{array}{llll}
\text{lemma2 O} & \text{O} & \_ & := () \\
\text{lemma2 (S } n) & (\text{S } m) & H & := \text{lemma2 } n\ m\ H \\
\text{lemma2 O} & (\text{S } m) & H & := \text{abort } H \\
\text{lemma2 (S } n) & \text{O} & H & := \text{abort } H
\end{array}
$$

Real Numbers

Approximations

Real Numbers

$$f : \mathbb{Q}^{+} \Rightarrow \mathbb{Q}$$

Real Numbers

$$\pi \; 0.01 \rightarrow 3.14$$

Well Behaved Approximations

Real Numbers

$$|f\,\delta - f\,\varepsilon| \leq \delta + \varepsilon$$

Real Numbers

$$\mathbb{R} := \Sigma f{:}\mathbb{Q}^+ \Rightarrow \mathbb{Q}.\ \forall \delta\ \varepsilon{:}\mathbb{Q}^+.\ \langle |f\,\delta - f\,\varepsilon| \leq \delta + \varepsilon \rangle$$

Real Numbers

$\iota : \mathbb{Q} \Rightarrow \mathbb{R}$

$\iota : \mathbb{Q} \Rightarrow \sum f{:}\mathbb{Q}^{+}{\Rightarrow}\mathbb{Q}.\ \forall\delta\ \varepsilon{:}\mathbb{Q}^{+}.\ \langle |f\,\delta - f\,\varepsilon| \leq \delta + \varepsilon \rangle$

$\iota\ q := (\lambda\_.\ q,\ \lambda\delta\ \varepsilon.\ \text{lemma3}\ q\ (\delta{+}\varepsilon))$

where

$\text{lemma3} : \forall q{:}\mathbb{Q}.\ \forall\varepsilon{:}\mathbb{Q}^{+}.\ \langle |q - q| \leq \varepsilon \rangle$

Uniformly Continuous Functions

$$\mathbb{Q} \rightarrow_{uc} \mathbb{Q} :=$$

$$\textstyle\sum f{:}\mathbb{Q}{\Rightarrow}\mathbb{Q}.\ \forall \varepsilon{:}\mathbb{Q}^{+}.\ \exists \delta{:}\mathbb{Q}^{+}.\forall p,q{:}\mathbb{Q}.$$

$$\langle |p - q| \leq \delta \rangle \Rightarrow \langle |f\,p - f\,q| \leq \varepsilon \rangle$$

Uniformly Continuous Functions

$$\mu : (\mathbb{Q} \rightarrow_{uc} \mathbb{Q}) \Rightarrow \mathbb{Q}^{+} \Rightarrow \mathbb{Q}^{+}$$

$$\mu\ (\_, p)\ \varepsilon := \pi_1\ (p\ \varepsilon)$$

Uniformly Continuous Functions

$$\mathrm{lift} : (\mathbb{Q} \to_{uc} \mathbb{Q}) \Rightarrow \mathbb{R} \Rightarrow \mathbb{R}$$

$$\mathrm{lift}\, f\, x := (\pi_1 f \circ x \circ \mu f, \ldots)$$

Uniformly Continuous Functions

$$(-_{\mathbb{R}}) := \mathrm{lift}\,(-_{\mathrm{uc}})$$

Uniformly Continuous Functions

$$\mathrm{lift2} : (\mathbb{Q} \to_{uc} \mathbb{Q} \to_{uc} \mathbb{Q}) \Rightarrow \mathbb{R} \Rightarrow \mathbb{R} \Rightarrow \mathbb{R}$$

Uniformly Continuous Functions

$$(+_{\mathbb{R}}) := \mathrm{lift2}\ (+_{\mathrm{uc}})$$

Uniformly Continuous Functions

$$(\min_{\mathbb{R}}) := \text{lift2}\ (\min_{\text{uc}})$$

Uniformly Continuous Functions

$$(\max_{\mathbb{R}}) := \text{lift2}\ (\max_{uc})$$

Works

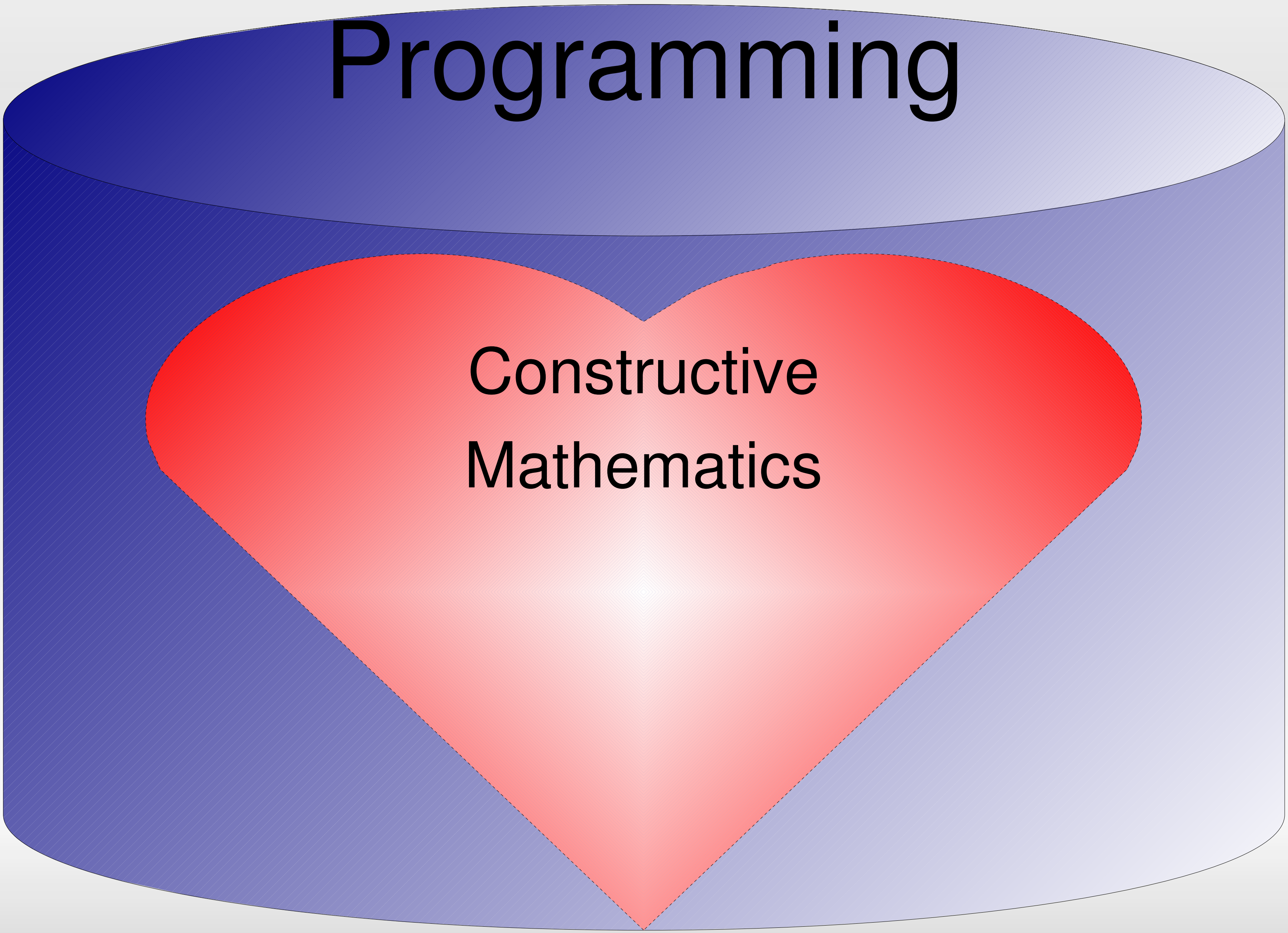
Mathematics

Computes

Real Numbers

$$(\pi_1\ \pi)\ 0.01 \rightarrow 805/256$$

Constructive Mathematics



Classical Logic

$$A \vee_c B := \neg(\neg A \wedge \neg B)$$
$$\exists_c a{:}A.\ (B\ a) := \neg\forall a{:}A.\ \neg(B\ a)$$

Law of the Excluded Middle

$$\lambda(f, g).\ g\,f : A \vee_{\mathrm{c}} \neg A$$

Classical Mathematics